Exercise

$\alpha \in (0,1)$. For every $E \subset \mathbb{R}$ not bounded above, $\sup E = +\infty$.

For every mapping $f : [0,+\infty) \to \mathbb{R}$, define

$$\|f\|_\alpha = \sup_{t \geq 0} \{ |e^{-2\alpha t} f(t)| \}.$$

Let $V$ be the set of $f : [0,+\infty) \to \mathbb{R}$ such that $\|f\|$ is finite

1) Show that for $f(t) = t$, $t \geq 0$, $\|f\|_\alpha$ is finite.

$$\|f\|_\alpha = \sup_{t \geq 0} \{ |e^{-2\alpha t} t| \}$$

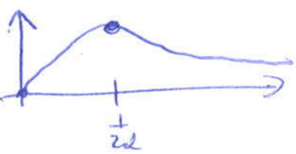
It's straightforward that $\varphi(t) = |e^{-2\alpha t} t| = e^{-2\alpha t} t$ has a maximum;

$$\varphi'(t) = e^{-2\alpha t} + e^{-2\alpha t} \cdot (-2\alpha t) = 0 \iff$$

$$\iff 1 - 2\alpha t = 0 \iff \boxed{\bar{t} = \frac{1}{2\alpha}}.$$

$$\varphi(\bar{t}) = e^{-1} \cdot \frac{1}{2\alpha} = \frac{1}{2\alpha e}.$$

$$\underset{(<)}{\varphi'(t) > 0} \iff \underset{(>)}{t < \frac{1}{2\alpha}}$$




Hence, $\|f\|_\alpha = \frac{1}{2\alpha e} < \infty$, so $f \in V$.

2) Prove that $(V, \|\cdot\|)$ is a normed vector space.

a) $\|f\|_\alpha = 0 \iff \sup_{t \geq 0} \{ |e^{-2\alpha t} f(t)| \} = 0 \iff f(t) = 0 \;\; \forall t \geq 0 \iff f = 0$.

b) $\|kf\|_\alpha = \sup_{t \geq 0} \{ |e^{-2\alpha t} k f(t)| \} = |k| \cdot \|f\|_\alpha$.

c) $\|f+g\|_\alpha \overset{?}{\leq} \|f\|_\alpha + \|g\|_\alpha$

$$\|f+g\|_\alpha = \sup_{t \geq 0} \{ |e^{-2\alpha t}(f(t)+g(t))| \} \leq \sup_{t \geq 0} \{ |e^{-2\alpha t} f(t)| + |e^{-2\alpha t} g(t)| \}$$

$$\leq \|f\|_\alpha + \|g\|_\alpha \qquad \square$$

3) $(f_n)$ a Cauchy sequence in $(V, \|\cdot\|)$

$\forall_{\varepsilon>0} \exists_{\tilde{n}} \forall_{m,n>\tilde{n}} \|f_n - f_m\|_\alpha < \varepsilon$

a) If $\sup_{t\geq 0} |(f_n(t) - f_m(t))e^{-2\alpha t}| < \varepsilon$

$\Rightarrow \forall_{t\geq 0} |f_n(t) - f_m(t)| < \varepsilon \cdot e^{2\alpha t} := \delta$

For all $t \geq 0$ fixed, one can always find $\tilde{n}$ that for $m, n > \tilde{n}$, it's ok:

$\forall_{\delta>0} \exists_{\tilde{n}} \forall_{m,n>\tilde{n}} |f_n(t) - f_m(t)| < \delta$

so $(f_n(t))$ is a Cauchy sequence.

b) $\boxed{\text{Banach space}}$ = normed and $\boxed{\text{complete}}$ vector space.

Every Cauchy sequence should converge to a limit in $V$.

Completeness of $\mathbb{R}$ is known. Hence for all $t \geq 0$,

$(f_n(t))$ – Cauchy, converges to $f(t)$.

Hence the only candidate for $f := \lim_{n\to\infty}(f_n)$ is the pointwise limit $f(t) := \lim_{n\to\infty} f_n(t)$, $\forall t \geq 0$.

We have: $\forall_{t\geq 0} \forall_{\varepsilon>0} \exists_{\tilde{n}} \forall_{n>\tilde{n}} \underbrace{|f_n(t) - f(t)| e^{-2\alpha t}}_{<\varepsilon} < \varepsilon$

and thus $\sup_{t\geq 0} |e^{-2\alpha t}(f_n(t) - f(t))| < \varepsilon$.

Is the limit continuous? (All $f \in V$ have to be.)

Let $t, u \geq 0$, $\|t-u\| < \delta$.

$\forall_{\varepsilon>0} \exists \delta \; \|t-u\| < \delta \Rightarrow \|f(t) - f(u)\| \leq \|f(t) - f_n(t)\|$
$+ \underbrace{\|f_n(t) - f_n(u)\|}_{\text{continuous}} + \|f_n(u) - f(u)\| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \; \square$

Let $F:\mathbb{R}\to\mathbb{R}$ be Lipschitz; $\forall t,u\geq 0$

$$|F(t)-F(u)| \leq \alpha \cdot \| t-u \|.$$

Define for $f\in V$, $T(f):[0,+\infty)\to\mathbb{R}$, $T: V\to V$.

Let $\forall t\geq 0$ $\quad T(f)(t) = y_0 + \int_0^t F(f(s))\,ds$

5) $T(0)(t) = y_0 + \int_0^t F(0)\,ds = y_0 + F(0)\cdot t := \varphi(t)$

Clearly, $\|\varphi(t)\| = \sup\limits_{t\geq 0} \left\{ \left| e^{-2\alpha t}(y_0 + F(0)t) \right| \right\} \leq$

$$\leq \underbrace{\sup_{t\geq 0} \{ | e^{-2\alpha t} t | \}}_{\frac{1}{2\alpha e}} \cdot F(0) + y_0 < \infty$$

so $\varphi \in V$.

6) $|T(f)(t) - T(g)(t)| = \left| \int_0^t F(f(s)) - F(g(s))\,ds \right|$

$$\underset{\substack{\uparrow\\ \text{triangle}\\ \text{ineq}}}{\leq} \int_0^t |F(f(s)) - F(g(s))|\,ds \underset{\substack{\uparrow\\ \text{Lipschitz}}}{\leq} \alpha \int_0^t |f(s)-g(s)|\,ds$$

$$\leq \alpha \int_0^t \underbrace{|f(s)-g(s)| e^{-2\alpha s}} \cdot e^{2\alpha s}\,ds \leq$$

$$\leq \alpha \sup_{s\in[0,t]} \{ |f(s)-g(s)| e^{-2\alpha s} \} \cdot \underbrace{\int_0^t e^{2\alpha s}\,ds}_{\frac{1}{2\alpha} e^{2\alpha s}\big|_0^t = \frac{1}{2\alpha}(e^{2\alpha t}-1)} \leq$$

$$\leq \frac{e^{2\alpha t}}{2} \| f-g \|_\alpha .$$

7) Deduce that $\forall f \in V$, $T(f) \in V$.



$$\|f\| < \infty \Rightarrow \|T(f)\| = \sup_{t \geq 0} \{ |T(f)(t) e^{-2\alpha t}| \}$$

We know

$$\|f\| = \|f-0\| \Rightarrow \|T(f) - T(0)\| = \sup_{t \geq 0} \left| \int_0^t (F(f(s)) - F(0)) \, ds \right| e^{-2\alpha t} \leq$$

$$\leq \sup_{t \geq 0} \left\{ e^{-2\alpha t} \cdot \frac{e^{2\alpha t}}{2} \cdot \|f - 0\| \right\} = \frac{1}{2} \|f\|$$

Since $\|T(0)\| < \infty$ and ~~$\|T(f) - T(0)\| \leq \|T(f)\| + \|T(0)\| < \infty$~~

$$\|T(f) + 0\| \leq \|T(f) - T(0)\| + \|T(0)\| < \infty.$$

Hence $T(f) \in V$.

8) $$\|T(f) - T(g)\| = \sup_{t \geq 0} \left\{ \left| e^{-2\alpha t} \int_0^t F(f(s)) - F(g(s)) \, ds \right| \right\}$$

$$= \sup_{t \geq 0} \left\{ (e^{-2\alpha t}) \cdot |T(f)(t) - T(g)(t)| \right\} \leq$$

$$\leq \sup_{t \geq 0} \left\{ e^{-2\alpha t} \cdot \frac{e^{2\alpha t}}{2} \|f - g\| \right\} = \frac{1}{2} \|f - g\|$$

Hence, $T$ is a contraction, with $\beta = 1/2$.

9) By Banach's Contraction Mapping Theorem, $T$ has a unique fixed point in the Banach space $V$ and thus $\exists ! \tilde{f} \in V$ s.t. $T(\tilde{f}) = \tilde{f}$.

<u>Exercise</u>



$(E, d)$ – a complete metric space

$\Lambda$ – metric space

$f : E \times \Lambda \to E$

- $\forall x \in E \quad g(\lambda) := f(x, \lambda)$ is continuous
- $\exists k < 1 \quad d(f(x,\lambda), f(y,\lambda)) \leq k\, d(x,y) \quad \forall \lambda \in \Lambda \ \forall_{x,y \in E}$

(contraction wrt. x)

a) Show that $\forall \lambda \in \Lambda \ \exists!\, a_\lambda \in E$ such that $f(a_\lambda, \lambda) = a_\lambda$.

$\forall \lambda \in \Lambda \quad \varphi_\lambda(x) := f(x, \lambda)$ is a contraction.

By Banach's theorem, $\varphi_\lambda(x)$ has a unique fixed point, $a_\lambda$.

Also, <u>$\varphi_\lambda(x)$ is continuous, $\forall \lambda \in \Lambda$</u>

b) $\psi(\lambda) = a_\lambda$

Take $\lambda_1, \lambda_2 \in \Lambda$, with $d(\lambda_1, \lambda_2) < \delta$

Then $d(a_{\lambda_1}, a_{\lambda_2}) = d(f(a_{\lambda_1}, \lambda_1), f(a_{\lambda_2}, \lambda_2)) \leq$

$$\leq \underbrace{d(f(a_{\lambda_1}, \lambda_2), f(a_{\lambda_1}, \lambda_1))}_{\text{CONT. OF } g(\lambda)} + \underbrace{d(f(a_{\lambda_2}, \lambda_2), f(a_{\lambda_1}, \lambda_2))}_{\text{CONT. OF } \varphi_{\lambda_2}(x)}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

Exercise $(X,d)$ – nonempty, compact metric space (35)

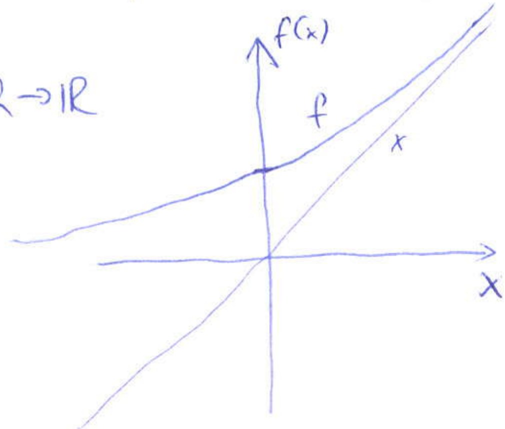
$f: X \to X$ satisfies $d(f(x), f(y)) < d(x,y)$ $\forall x,y \in X$, $x \neq y$.

a) this is a weakly contracting mapping — difference may converge to 0.

Under compactness, it still has a unique fixed point $x^* \in X$.

b) Without compactness, it's not true.

e.g. $f: \mathbb{R} \to \mathbb{R}$




$|f(x) - f(y)| < |x-y|$ $\forall x,y \in \mathbb{R}$

Let $x > y$, $x - y := h$

$|f(y+h) - f(y)| < h$

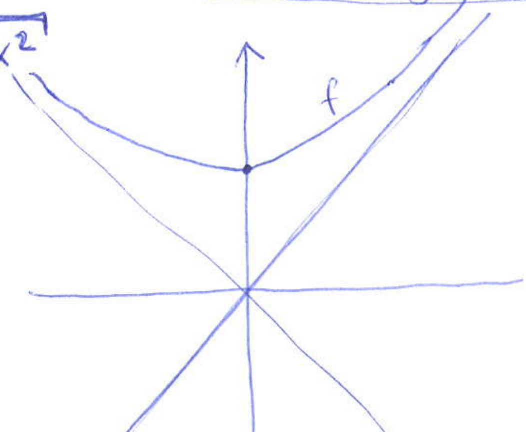
$\frac{|f(y+h) - f(y)|}{h} < 1$

Taking $\lim_{h \to 0}$, we have $|f'(y)| < 1$ for any differentiable $f: \mathbb{R} \to \mathbb{R}$

(Exact counterexample)

e.g. $f(x) = \sqrt{1+x^2}$

$f'(x) = \frac{2x}{2\sqrt{1+x^2}}$




f – weakly contracting but not a contraction !

f is asymptotic to $y = x$ in $+\infty$ & $y = -x$ in $-\infty$

$f(x) = x \iff x = \sqrt{1+x^2}$

$\iff 1 = \sqrt{\frac{1+x^2}{x^2}} \iff 1 = \sqrt{1 + \underbrace{\frac{1}{x^2}}_{>0}}$ contradiction

if $x \neq 0$

$x = 0 \Rightarrow$ contradiction as well.

Solving point (a).

Note This may be not the simplest possible proof. At least I am sure that it solves the problem.

First, define a sequence $\{x_n\}$, where $x_0$ - given, $x_1 = f(x_0), \dots, x_n = f^n(x_0)$ (and thus $x_2 = f(x_1)$, $x_{n+1} = f(x_n)$, etc.)

By weak contraction property, $d(x_{n+1}, x_n) < d(x_n, x_{n-1}) < \dots < d(x_1, x_0)$.

Second, define a sequence $\{\varphi_n\} \subset X \times X$, where $X \times X$ is endowed with the product metric:

$$d(\varphi_n, \varphi_m) = \sqrt{d^2(\varphi_{n1}, \varphi_{m1}) + d^2(\varphi_{n2}, \varphi_{m2})}.$$

Define $\varphi_n = (x_{n+1}, x_n)$ $\forall n \in \mathbb{N}$. As $X$ - compact, we know that $X \times X$ - compact as well. Hence, $\{\varphi_n\}$ must have a convergent subsequence $\{\varphi_{n_k}\}$ such that $\varphi_{n_k} \xrightarrow{k \to \infty} \tilde{\varphi}$, with $\tilde{\varphi} \in X \times X$.

Let $\tilde{\varphi} := (x_1^*, x_2^*)$. We have (due to convergence of $\varphi_{n_k}$):

$$\forall \varepsilon > 0 \ \exists \tilde{k} \in \mathbb{N} \ \forall k > \tilde{k} \quad d(\varphi_{n_k}, \tilde{\varphi}) = \sqrt{d^2(x_{n_{k+1}}, x_1^*) + d^2(x_{n_k}, x_2^*)} < \frac{\varepsilon}{3}.$$

Hence, $d(x_{n_{k+1}}, x_1^*) < \frac{\varepsilon}{3}$

$d(x_{n_k}, x_2^*) < \frac{\varepsilon}{3}$

and also $d(x_{n_{k+1}}, x_{n_k}) < \frac{\varepsilon}{3}$ (because $\{x_{n_k}\}$ is convergent as well).

Hence $\forall \varepsilon > 0$ $d(x_1^*, x_2^*) \underset{\substack{\uparrow \\ \text{triangle} \\ \text{ineq}}}{\leq} d(x_1^*, x_{n_{k+1}}) + d(x_{n_{k+1}}, x_{n_k}) + d(x_{n_k}, x_2^*)$

$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$

Therefore $d(x_1^*, x_2^*) = 0 \Rightarrow x_1^* = x_2^*$. By definition of $\{\varphi_n\}$, $\underline{\underline{x_1^* = x_2^*}}$ is a fixed point of $f$, $x_1^* = f(x_1^*)$.

To show that $x_1^*$ is unique, assume that it's not (proof by contradiction). Let $x', x'' \in X$ be different fixed points. Then $d(x', x'') \underset{\substack{\uparrow \\ \text{fixed} \\ \text{points}}}{=} d(f(x'), f(x'')) \underset{\substack{\uparrow \\ \text{weak} \\ \text{contraction}}}{<} d(x', x'')$. Hence we get a contradiction and thus $x' = x''$.

□